On the mathematical connections between some equations concerning the calculation of all the eigenfunctions of atoms with the Thomas-Fermi method, some sectors of Number Theory, the modes corresponding to the physical vibrations of superstrings, p-Adic and Adelic free relativistic particle and p-Adic strings.

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Abstract

According to quantum mechanics, the properties of an atom can be calculated easily if we known the eigenfunctions and eigenvalues of quantum states in which the atom can be found. The eigenfunctions depend, in general, by the coordinates of all the electrons. However, a diagram effective and enough in many cases, we can get considering the individual eigenfunctions for individual electrons, imagining that each of them is isolated in an appropriate potential field that represent the action of the nucleus and of other electrons. From these individual eigenfunctions we can to obtain the eigenfunction of the quantum state of the atom, forming the antisymmetrical products of eigenfunctions of the individual quantum states involved in the configuration considered. The problem, with this diagram, is the calculation of the eigenfunctions and eigenvalues of individual electrons of each atomic species. To solve this problem we must find solutions to the Schroedinger's equation where explicitly there is the potential acting on the electron in question, due to the action of the nucleus and of all the other electrons of the atom. To research of potential it is possible proceed with varying degrees of approximation: a first degree is obtained by the statistical method of Thomas-Fermi in which electrons are considered as a degenerate gas in balance as a result of nuclear attraction. This method has the advantage of a great simplicity as that, through a single function numerically calculated once and for all, it is possible to represent the behaviour of all atoms. In this work (**Sections 1** and **2**) we give the preference to the statistical method, because in any case it provides the basis for more approximate numerical calculations. Furthermore, we describe the mathematical connections that we have obtained between certain solutions concerning the calculation of any eigenfunctions of atoms with this method, the Aurea ratio, the Fibonacci's numbers, the Ramanujan modular equations, the modes corresponding to the physical vibrations of strings, the p-adic and Adelic free relativistic particle and p-adic and adelic strings (**Sections 3** and **4**).

1. Calculation of potential. [1]

The considerations that we can do in this paragraph refer to more general case of an atomic number Z, ionized z times. To establish the differential equation that determines the potential with the distance from the nucleus, it shall by the relation that connects the density of electronic gas with a potential if the electrons can be treated as a completely degenerate gas. This relation is

$$n = \frac{2^{9/2}\pi m^{3/2}e^{3/2}}{3h^3}(V+\alpha)^{3/2} \quad \text{for} \quad V+\alpha>0; \qquad n=0 \quad \text{for} \quad V+\alpha<0 \qquad (1)$$

α is an additive constant that can be determined with the condition that the total number of electrons is that given and that is

$$\int n\,d\tau = Z - z$$

where the integral should be extended to the whole region of the space where $n \neq 0$. It must be borne in mind that the potential on a electron does not coincide exactly with the potential that we have in a geometric point that is at an equal distance from the nucleus. In fact, the first potential represents the action of the nucleus and of $Z-z-1$ electrons, while the second is the action of the nucleus and of all the $Z-z$ electrons. We will denote with V and V' the two potentials now defined. We will have:

$$V = \frac{Ze}{r} + U\,, \qquad V' = \frac{Ze}{r} + U'$$

where U and U' represents respectively the actions of the $Z-z-1$ and $Z-z$ electrons. We consider therefore, in first approximation, U and U' respectively proportional to $Z-z-1$ and $Z-z$, thence we will write

$$U' = \frac{Z-z}{Z-z-1}U\,.$$

V' is the potential due to the nucleus and all the electrons: therefore for it we have the Poisson's equation

$$\Delta V' = \Delta U' = 4\pi ne$$

and we obtain

$$\Delta V = \Delta U = \frac{Z-z-1}{Z-z}\Delta U' = \left(1-\frac{1}{Z-z}\right)4\pi ne\,.$$

Taking account of the (1) and the fact that α is constant and by placing

$$v = V + \alpha$$

we obtain

$$\Delta v = \left(1-\frac{1}{Z-z}\right)\frac{2^{13/2}\pi^2 m^{3/2}e^{5/2}}{3h^3}v^{3/2} \quad \text{for} \quad v>0; \qquad \Delta v = 0 \quad \text{for} \quad v<0\,.$$

Because v for reasons of symmetry depends only from r, the previous equation becomes

$$\frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}=\left(1-\frac{1}{Z-z}\right)\frac{2^{13/2}\pi^2 m^{3/2}e^{5/2}}{3h^3}v^{3/2} \quad \text{for} \quad v>0; \qquad \frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}=0 \quad \text{for} \quad v<0.$$

The constants of integration can be determined with the conditions

$$\lim_{r=0} vr = Ze \qquad \int_0^{r_0} 4\pi r^2 n dr = Z-z$$

where r_0 is the distance for which v is null, the distance to which is to end the electronic cloud surrounding the nucleus. To simplify the previous equations, we put

$$r=\mu x$$

where

$$\mu=\frac{3^{2/3}h^2}{2^{13/3}\pi^{4/3}me^2}\frac{(Z-z)^{2/3}}{Z^{1/3}(Z-z-1)^{2/3}}=\left(\frac{9\pi^2}{128}\right)^{1/3} a\frac{(Z-z)^{2/3}}{Z^{1/3}(Z-z-1)^{2/3}} \quad (2)$$

being a the radius of the first orbit of the hydrogen,

$$v=\frac{eZ}{\mu}\frac{\varphi(x)}{x}. \quad (3)$$

We note that $\sqrt[3]{\frac{9\pi^2}{128}}=0,885341377$, and for $(\Phi)^{n/7}$, we have that

$$0,885\cong 0,8917=(\Phi)^{-1,67/7}=\left(\frac{\sqrt{5}+1}{2}\right)^{-1,67/7}.$$

We have also that:

$(\Phi)^{-63/7}=0,0131556175\cong 0,013156$; $(\Phi)^{-49/7}=0,0344418537\cong 0,034442$;
$(\Phi)^{-7/7}=0,6180339887\cong 0,618034$.
0,013156 + 0,034442 + 0,618034 = 0,665632; $\frac{4}{3}(0,665632)=\mathbf{0,887509}$.

$(\Phi)^{-35/7}=0,0901699437\cong 0,090170$; $(\Phi)^{-21/7}=0,2360679775\cong 0,236068$; $(\Phi)^0=1$.
0,090170 + 0,236068 + 1 = 1,326238; $\frac{2}{3}(1,326238)=\mathbf{0,884159}$.

It's interesting the observation that these values coincide almost to those given, i.e. 0,885341, and that the index 63, 49, 35 and 21 are all multiples of 7.
We note that, **with regard the formula $(\Phi)^{n/7}$, Φ represent the Aurea Ratio, n a real number (positive or negative) and 7 are the compactified dimensions of M-theory**.

Thence, we obtain for the function $\varphi(x)$ the differential equation

$$\varphi'' = \frac{\varphi^{3/2}}{\sqrt{x}} \quad (4)$$

with the boundary conditions

$$\varphi(0) = 1 \qquad \int_0^{x_0} \varphi^{3/2} \cdot \sqrt{x}dx = 1 - \frac{z+1}{Z} \quad (5)$$

where $x_0 = \frac{r_0}{\mu}$ and $\varphi(x_0) = 0$. The last integral can be made taking into account of the (4) and we found that it is equivalent to

$$-x_0\varphi'(x_0) = \frac{z+1}{Z}. \quad (6)$$

One finds that in widely sufficient approximation for the practical cases we can set

$$\varphi = \varphi_0 - k\eta \quad \text{for} \quad x > x_0; \qquad \varphi = -\frac{z+1}{Z}\frac{x-x_0}{x_0} \quad \text{for} \quad x < x_0. \quad (7)$$

Here k is a constant the value of which depends on $(z+1)/Z$ in order to meet the (6). An empirical expression that represents k with sufficient accuracy within the interval of values that interested, is the following

$$k = 0.083\left(\frac{z+1}{Z}\right)^3. \quad (8)$$

We note that for $\left(\frac{\sqrt{5}+1}{2}\right)^{n/7} = (\Phi)^{n/7} = (\Phi)^{-36/7} = 0{,}08417 \cong 0{,}083$. Thence, from the eq. (8), we obtain

$$k = (\Phi)^{-36/7}\left(\frac{z+1}{Z}\right)^3.$$

We have also that

$(\Phi)^{-56/7} = 0{,}0212862363 \cong 0{,}021286$; $\quad (\Phi)^{-28/7} = 0{,}1458980338 \cong 0{,}145898$.

0,021286 + 0,145898 = 0,167184; $\quad \frac{1}{2}(0{,}167184) = \mathbf{0{,}083592}$.

$(\Phi)^{-56/7} = 0{,}0212862363 \cong 0{,}021286$; $\quad (\Phi)^{-35/7} = 0{,}0901699437 \cong 0{,}090170$.

0,021286 + 0,090170 = 0,111456; $\quad \frac{3}{4}(0{,}111456) = \mathbf{0{,}083592}$.

We note that these values coincide almost to those given, i.e. 0,083.

The potential v given by (3) isn't null to infinity: it is therefore appropriate to add it a constant so as to aim to zero for $x \to \infty$. It is recognised immediately that this constant has the value

$$\frac{e}{\mu}\frac{z+1}{x_0}.$$

We will take therefore in the final analysis as an expression of the potential the following

$$V=\frac{e}{\mu}\left[\frac{Z}{x}\varphi(x)+\frac{z+1}{x_0}\right]. \quad (9)$$

2. The relativistic equations of quantum mechanics. [1]

Established the potential of the strength field in which the electron in question moves, we consider the relativistic equations of quantum mechanics. Thence, in this section we describe various equations concerning ***the free relativistic electron.***
We consider the terms $s(l=0)$ it is possible write these equations in the following form

$$\frac{2\pi}{h}\left(\frac{W-eV}{c}+2mc\right)F+\frac{dG}{dr}=0 \qquad -\frac{2\pi}{h}\frac{W-eV}{c}G+\frac{dF}{dr}+\frac{2}{r}F=0 \quad (10)$$

where F and G are the eigenfunctions, W the energy and V the potential. The eigenfunctions F and G are connected to the four Dirac's functions from the relations

$$\begin{cases}\psi_1=-iF\cos\theta\\ \psi_2=-iF\sin\theta e^{i\varphi}\\ \psi_3=G\\ \psi_4=0\end{cases} \qquad \begin{cases}\psi_1=-iF\sin\theta e^{-i\varphi}\\ \psi_2=iF\cos\theta\\ \psi_3=0\\ \psi_4=G.\end{cases} \qquad (11)$$

Putting in the eq. (10) the eq. (9) to the place of V, and

$$W=-\frac{e^2}{2a}\varepsilon$$

one obtain

$$\left[\frac{2}{\alpha}\frac{\mu}{a}-\frac{\alpha}{2}\frac{\mu}{a}\varepsilon+\alpha Z\left(\frac{\varphi}{x}+\frac{1}{Zx_0}\right)\right]F+G'=0\,;\ \left[\frac{\alpha}{2}\frac{\mu}{a}\varepsilon-\alpha Z\left(\frac{\varphi}{x}+\frac{1}{Zx_0}\right)\right]G+F'+\frac{2}{x}F=0. \quad (12)$$

In these equations we have introduced the variable $x=\frac{r}{\mu}$: α is the Fine Structure Constant that has the numerical value **1/137.3**.
We note that multiplying the frequency 306,342224 for $(\Phi)^{5/7}=1,410187582$ (where $\Phi=\frac{\sqrt{5}+1}{2}$), we obtain the frequency 432. Dividing this frequency for π , we obtain **137.5** that is a value very near to the inverse of **Fine Structure Constant**. Furthermore, dividing 432 for Φ and Φ^2, we

obtain the numbers 267 and 165. The sum of these numbers, provide again 432. We note also that the numbers 267 and 165 are given by sums of Fibonacci's numbers. Indeed, we have

$$267 = 233 + 34 \quad \text{and} \quad 165 = 144 + 21 \qquad (233 = 89 + 144; \quad 144 = 55 + 89),$$

where 21, 34, 55, 89, 144 and 233 are Fibonacci's numbers.
Furthermore, we can to obtain the inverse value of Fine Structure Constant also from the Aura ratio. Indeed, we have:

$$(\Phi)^{35/7} = 11{,}0901699437\,; \quad (\Phi)^{21/7} = 4{,}2360679775\,; \quad (\Phi)^{7/7} = 1{,}6180339887\,;$$

$$11{,}090170 + 4{,}236068 + 1{,}618034 = 16{,}944272;$$

$$\frac{3}{2}(16{,}944272) = 25{,}416408\,; \quad 2(16{,}944272) = 33{,}888544\,; \quad 3(16{,}944272) = 50{,}832816\,;$$

$$1{,}61803398875(16{,}944272) = 27{,}416408\,;$$

$$25{,}416408 + 33{,}888544 + 50{,}832816 + 27{,}416408 = \mathbf{137{,}554176}.$$

We note also that dividing 432 for this value, we obtain a value very near to π. Indeed, we have:

$$432/137{,}554176 = 3{,}14058077 \cong \pi$$

Given that the function $\varphi_0(x)$, through which we express the statistical potential, is represented with sufficient accuracy to the practical purposes, for $x \leq 0.3$, by the following empirical expression

$$\varphi_0(x) = 1 - px + qx^2 = 1 - 1.304x + 1.288x^2\,, \quad (13)$$

We can try to meet the equations (12) within the interval $x = 0 \quad x = 0.3$ by developments in power series of x. For $x > 0.3$ we will try instead the solutions of equations (12) with the Wentzel-Brillouin's method. This method is used for the execution of calculations in series because it replaces to a irregular function, two functions of regular development: the amplitude and the phase; the interpolations must run on the latter. Now is appropriate to modify a little about the form in which have been written the equations (12). Putting

$$F = Z\alpha\mathcal{R}$$

where $\mathcal{R}$ is the new function unknown, thence we have

$$\left[\frac{2\mu Z}{a} - \frac{\alpha^2}{2}\frac{\mu Z}{a}\varepsilon + \alpha^2 Z^2\left(\frac{\varphi}{x} + \frac{1}{Zx_0}\right)\right]\mathcal{R} + G' = 0\,; \quad \left[\frac{\mu}{2aZ}\varepsilon - \left(\frac{\varphi}{x} + \frac{1}{Zx_0}\right)\right]G + \mathcal{R}' + \frac{2}{x}\mathcal{R} = 0 \quad (14)$$

from these, remembering the (13), one obtain

$$\left\{\frac{a}{2\mu Z} - \alpha^2 Z^2\left[\frac{\mu\varepsilon}{2\alpha Z} - \left(\frac{1}{x} - p + \frac{1}{Zx_0} + qx\right)\right]\right\}\mathcal{R} + G' = 0\,;$$

$$\left[\frac{\mu}{2a}\frac{\varepsilon}{Z}-\left(\frac{1}{x}-p+\frac{1}{Zx_0}+qx\right)\right]G+\mathcal{R}'+\frac{2}{x}\mathcal{R}=0\,. \quad (15)$$

If we put, for simplicity

$$T=p-\frac{1}{Zx_0}+\frac{\mu\varepsilon}{2aZ}$$

we obtain

$$\mathcal{R}'+\frac{2}{x}\mathcal{R}-\left(\frac{1}{x}-T+qx\right)G=0\,;\quad G'+\left[\frac{2\mu Z}{a}+\alpha^2Z^2\left(\frac{1}{x}-T+qx\right)\right]\mathcal{R}=0\,. \quad (16)$$

In these equations G and R are equal at infinity in the initial point. To eliminate this singularity we put:

$$G=x^{-\beta}u\,,\qquad \mathcal{R}=\frac{\beta}{\alpha^2Z^2}x^{-\beta}v\,. \quad (17)$$

By introducing these expressions in the eqs. (16) and by accepting that u and v are regular for $x=0$, we obtain for β an equation of second degree :

$$\beta(2-\beta)=\alpha^2Z^2\,.$$

Of the two possible values of β is easily seen that one only meets to our purpose because it must be less than 3/2, so that G^2, that in the initial point is behaving as $x^{-2\beta}$, can be integrable for $x=0$. From this we have that:

$$\beta^2-2\beta+\alpha^2Z^2=0\,;\quad \beta_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{2\pm\sqrt{4-4\alpha^2}Z^2}{2}=1\pm\sqrt{1-\alpha^2}Z^2\,;$$

and we take the following value: $\beta=1-\sqrt{1-\alpha^2}Z^2\,. \quad (18)$

We obtain in the final analysis the differential equations concerning the two new functions unknown u and v

$$-\frac{xv'}{2-\beta}-v+u-Txu+qx^2u=0\,;\qquad -\beta u+xu'+\beta v+Lxv+\beta qx^2v=0\,. \quad (19)$$

In these equations, we have put for simplicity

$$L=\frac{2\mu Z}{a(2-\beta)}-T\beta\,.$$

Now we put

$$\begin{cases} u=1+u_1x+u_2x^2+\ldots \\ v=1+v_1x+v_2x^2+\ldots \end{cases} \quad (20)$$

and introduce these developments in series in the equations (19). We obtain in this way of the recurrent formulas that allow the numeric calculation of the coefficients

$$u_n = \frac{(2-\beta)\beta}{n(2-2\beta+n)}\left[Tu_{n-1} - qu_{n-2}\right] + \frac{2-\beta+n}{n(2-2\beta+n)}\left[-Lv_{n-1} - \beta q v_{n-2}\right];$$
$$v_n = \frac{(2-\beta)(n-\beta)}{n(2-2\beta+n)}\left[Tu_{n-1} - qu_{n-2}\right] + \frac{2-\beta}{n(2-2\beta+n)}\left[-Lv_{n-1} - \beta q v_{n-2}\right]. \quad (21)$$

We shall now proceed to the calculation of the solutions of equations (10) for $x > 0.3$. If we put for simplicity

$$g = -\frac{\mu^2}{a^2}\varepsilon + \frac{2\mu Z}{a}\left(\frac{\varphi}{x} - \frac{1}{Zx_0}\right) \quad (22)$$

the eqs. (14) thence, can be write

$$\mathcal{R}' + \frac{2}{x}\mathcal{R} - \frac{a}{2\mu Z}gG = 0 \quad (23a) \qquad G' + \mathcal{R}\left[\frac{2\mu Z}{a} + \frac{aZ}{2\mu}\alpha^2 g\right] = 0 . \quad (23b)$$

If we only consider terms up to α^2 included, we will get from the (23b)

$$\mathcal{R} = -\frac{a}{2\mu Z}G'\left(1 - \frac{a^2\alpha^2}{4\mu^2}g\right).$$

If we insert this expression in the eq. (23a), we obtain

$$gG + \left(G'' + \frac{2}{x}G'\right)\left(1 - \frac{a^2\alpha^2}{4\mu^2}g\right) - \frac{a^2\alpha^2}{4\mu^2}g'G' = 0 .$$

If we put

$$xG = U$$

so that

$$G' = \frac{U'}{x} - \frac{U}{x^2}; \qquad U'' = x\left(G'' + \frac{2}{x}G'\right)$$

we obtain

$$gU + U''\left(1 - \frac{a^2\alpha^2}{4\mu^2}g\right) - \frac{a^2\alpha^2}{4\mu^2}g'\left(U' - \frac{U}{x}\right) = 0 .$$

If we only consider terms up to α^2 included, we will get from the precedent equation

$$U'' + gU = \gamma^2\left\{g'U' - \frac{g'}{x}U - g^2U\right\} \quad (25)$$

where

$$\gamma^2 = \frac{a^2\alpha^2}{4\mu^2}. \quad (26)$$

Now we are seeking the solution of the equation (25) with the method Wentzel-Brillouin. Since for the functions ∞s the function U have a trend oscillating for all the values of x, we put

$$U = e^{i\int Zdx} \quad (27)$$

where Z is the new unknown function. The part real and imaginary of U both are solutions of the equation. From the eq. (25) we obtain, by the eq. (27) the following equation

$$iZ' - Z^2 + g = \gamma^2\left\{igZ - \frac{g'}{x} + g^2\right\}. \quad (28)$$

In this equation the right hand side represent the relativistic correction. It can be solved by successive approximations: running the calculations until the fourth approximation, and taking for U the real expression:

$$U = K \cdot R\sin\Theta \quad (29)$$

we obtain

$$R(x) = \frac{1 + \dfrac{g''}{16g^2} - \dfrac{1}{64}\cdot 5\dfrac{g'^2}{g^3} + \dfrac{\gamma^2}{2}g}{\sqrt[4]{g}} \quad (30)$$

$$\Theta(x) = \Theta(0{,}3) + \left[\int\sqrt{g}\,dx - \frac{1}{32}\int\frac{g'^2}{g^{5/2}}dx - \frac{1}{8}\frac{g'}{g^{3/2}} + \frac{\gamma^2}{2}\left(\int\frac{g'dx}{x\sqrt{g}} + \int g^{3/2}dx\right)\right]_{0,3}^{x}. \quad (31)$$

We observe that K and $\Theta(0{,}3)$ are constants that must be determined so that for $x = 0.3$, the function $U(x)$ and its derivative is connect with continuity to the function $u(x)x^{1-\beta}$ determined with the development in series valid for $x < 0.3$. From the eq. (29), we obtain

$$K\sin\Theta = \frac{U}{R}; \quad K\cos\Theta = \frac{1}{R\Theta'}\left\{U' - U\frac{d\log R}{dx}\right\}. \quad (32)$$

Through the equations (30) and (31), can be calculated Θ', R and $\dfrac{d\log R}{dx}$ for $x = 0.3$ and the values thus obtained shall be introduced in the previous equations. In the case of eigenfunctions ∞s, being zero the eigenvalue, we can write the equations (30) and (31) with sufficient accuracy in the form

$$R = A^{-1/4}\left(\frac{x}{\varphi_0}\right)^{1/4}\left\{1 + \frac{1}{A}\left[\frac{1}{16}\left(\frac{x}{\varphi_0}\right)^{1/2} + \frac{1}{64}\cdot 3\frac{1}{x\varphi_0} - \frac{1}{64}\cdot 5\frac{x\varphi_0'^2}{\varphi_0^3} + \frac{1}{32}\frac{\varphi'_0}{\varphi_0^2}\right] + \frac{\gamma^2 A}{2}\frac{\varphi_0}{x}\right\} \quad (33)$$

$$\Theta(x)=\Theta(0,3)+A^{1/2}\left\{\int_{0,3}^{x}\left(\frac{\varphi_0}{x}\right)^{1/2}dx-\frac{k}{2}\int_{0,3}^{x}\frac{\eta}{(x\varphi_0)^{1/2}}dx+\frac{1}{2Zx_0}\int_{0,3}^{x}\left(\frac{x}{\varphi_0}\right)^{1/2}dx\right\}+$$

$$-A^{-1/2}\left\{\frac{1}{32}\left[\int_{0,3}^{x}\frac{x^{1/2}\varphi'^2_0}{\varphi_0^{5/2}}dx+\int_{0,3}^{x}\frac{dx}{\varphi_0^{1/2}x^{3/2}}-2\int_{0,3}^{x}\frac{\varphi'_0}{x^{1/2}\varphi_0^{3/2}}dx\right]+\frac{1}{8}\left(\frac{x^{1/2}\varphi'_0}{\varphi_0^{3/2}}-\frac{1}{(x\varphi_0)^{1/2}}\right)_{0,3}^{x}\right\}+$$

$$+\frac{\gamma^2}{2}\left\{A^{1/2}\left[\int_{0,3}^{x}\frac{\varphi'_0}{x^{3/2}\varphi_0^{1/2}}dx-\int_{0,3}^{x}\frac{\varphi_0^{1/2}}{x^{5/2}}dx\right]+A^{3/2}\int_{0,3}^{x}\left(\frac{\varphi_0}{x}\right)^{3/2}dx\right\}. \quad (34)$$

In these expressions the constant A depends on the atomic number and has the following value

$$A=2Z\frac{\mu}{a}.$$

Note that the number 8, and thence the numbers $64=8^2$ and $32=2^2\times 8$ contents in the equations (30), (31), (33) and (34), are connected with the "modes" that correspond to the physical vibrations of a superstring by the following Ramanujan function:

$$8=\frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (34a)$$

Furthermore, we note that is possible connect the eq. (34) also with the Palumbo-Nardelli model. In this model, we have the following general relationship that links the supersymmetric string action with the bosonic string action:

$$\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]=$$

$$=-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (34b)$$

Thence, we obtain the following interesting mathematical connection with $\Theta(x)$:

$$\Theta(x) = \Theta(0,3) + A^{1/2}\left\{\int_{0,3}^{x}\left(\frac{\varphi_0}{x}\right)^{1/2} dx - \frac{k}{2}\int_{0,3}^{x}\frac{\eta}{(x\varphi_0)^{1/2}} dx + \frac{1}{2Zx_0}\int_{0,3}^{x}\left(\frac{x}{\varphi_0}\right)^{1/2} dx\right\} +$$

$$- A^{-1/2}\left\{\frac{1}{32}\left[\int_{0,3}^{x}\frac{x^{1/2}\varphi'^2_0}{\varphi_0^{5/2}} dx + \int_{0,3}^{x}\frac{dx}{\varphi_0^{1/2}x^{3/2}} - 2\int_{0,3}^{x}\frac{\varphi'_0}{x^{1/2}\varphi_0^{3/2}} dx\right] + \frac{1}{8}\left(\frac{x^{1/2}\varphi'_0}{\varphi_0^{3/2}} - \frac{1}{(x\varphi_0)^{1/2}}\right)_{0,3}^{x}\right\} +$$

$$+ \frac{\gamma^2}{2}\left\{A^{1/2}\left[\int_{0,3}^{x}\frac{\varphi'_0}{x^{3/2}\varphi_0^{1/2}} dx - \int_{0,3}^{x}\frac{\varphi_0^{1/2}}{x^{5/2}} dx\right] + A^{3/2}\int_{0,3}^{x}\left(\frac{\varphi_0}{x}\right)^{3/2} dx\right\} \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2} e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_v\left(\left|F_2\right|^2\right)\right] =$$

$$= -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (34c)$$

The eqs. (33) and (34) are apply naturally only until a certain distance from the limit of the atom, precisely until η is small compared with φ_0; approaching the limit of the atom we must calculate the eigenfunctions ∞s element by element through the equations (30) and (31). Always in the case of eigenfunctions ∞s, because in the proximity of the origin we can certainly neglect the function η, it is possible to put for $x \le 0.3$

$$g = A\frac{\varphi_0}{x}.$$

Taking account of this simplification, the definitive formulas for the calculation of R, Θ' and $\frac{d\log R}{dx}$ for $x = 0.3$ are the following:

$$\begin{cases} R = \dfrac{0.804}{\sqrt[4]{A}}\left\{1 + \dfrac{0.190}{A} + 1.20\gamma^2 A\right\} \\ \Theta' = 1.55\sqrt{A} - \dfrac{0.588}{\sqrt{A}} + \dfrac{\gamma^2}{2}\left(3.73A^{3/2} - 2.60A^{1/2}\right) \\ \dfrac{d\log R}{dx} = 1.063 - \dfrac{0.519}{A} - 5.11\gamma^2 A. \end{cases} \quad (35)$$

In the following tables, we will give the numerical values of some expressions concerning the calculation of the functions R (eq. 33) and Θ (eq. 34), expressions that are valid only until a certain distance from the limit of the atom. We describe the mathematical connections between these numerical values and some powers of the Aurea Ratio, i.e. with $(\Phi)^{n/7}$.

x	$\frac{\varphi_0}{x}$	$(\Phi)^{n/7}$
0.3	2.40	$(\Phi)^{12.67/7} = 2.3887$
0.4	1.65	$(\Phi)^{7.33/7} = 1.6555$
0.5	1.21	$(\Phi)^{2.67/7} = 1.20119$
0.6	0.937	$(\Phi)^{-1/7} = 0.93356$
0.7	0.744	$(\Phi)^{-4.33/7} = 0.74238$
0.8	0.606	$(\Phi)^{-7.33/7} = 0.60403$
0.9	0.503	$(\Phi)^{-10/7} = 0.50285$
1.0	0.425	$(\Phi)^{-12.33/7} = 0.42833$
1.2	0.312	$(\Phi)^{-17/7} = 0.31078$
1.4	0.238	$(\Phi)^{-21/7} = 0.23606$
1.6	0.186	$(\Phi)^{-24.33/7} = 0.1877$
1.8	0.149	$(\Phi)^{-27.67/7} = 0.14927$
2.0	0.122	$(\Phi)^{-30.67/7} = 0.12146$

x	$\int_{0.3}^{x}\left(\frac{\varphi_0}{x}\right)^{1/2} dx$	$(\Phi)^{n/7}$
0.4	0.141	$(\Phi)^{-28.33/7} = 0.14259$
0.5	0.260	$(\Phi)^{-19.67/7} = 0.2587$
0.6	0.363	$(\Phi)^{-14.67/7} = 0.3648$
0.7	0.454	$(\Phi)^{-11.33/7} = 0.4588$
0.8	0.563	$(\Phi)^{-8.33/7} = 0.5639$
1.0	0.678	$(\Phi)^{-5.67/7} = 0.6773$
1.2	0.799	$(\Phi)^{-3.33/7} = 0.7952$
2.0	1.150	$(\Phi)^{2/7} = 1.14738$
5.0	1.771	$(\Phi)^{8.33/7} = 1.77335$
20	2.447	$(\Phi)^{13/7} = 2.4441$
24	2.504	$(\Phi)^{13.33/7} = 2.500758$

x	$\int_{0.3}^{x} \frac{\eta \cdot 10^{-4}}{(x\varphi_0)^{1/2}} dx$	$(\Phi)^{n/7}$
10	0.095	$(\Phi)^{-34.33/7} = 0.09439$
12	0.211	$(\Phi)^{-22.67/7} = 0.21051$
14	0.434	$(\Phi)^{-12/7} = 0.4382$
16	0.819	$(\Phi)^{-3/7} = 0.8136$
18	1.446	$(\Phi)^{5.33/7} = 1.4428$
20	2.447	$(\Phi)^{13/7} = 2.4441$
22	3.989	$(\Phi)^{20/7} = 3.9546$
24	6.247	$(\Phi)^{26.67/7} = 6.2537$
26	9.516	$(\Phi)^{32.67/7} = 9.4466$
28	14.057	$(\Phi)^{39/7} = 14.600$
30	20.356	$(\Phi)^{44/7} = 20.589$
32	28.867	$(\Phi)^{49/7} = 29.034$
34	40.136	$(\Phi)^{53.67/7} = 40.0256$

x	$\int_{0.3}^{x} \left(\frac{x}{\varphi_0}\right)^{1/2} dx$	$(\Phi)^{n/7}$
0.5	0.156	$(\Phi)^{-27/7} = 0.15628$
0.6	0.253	$(\Phi)^{-20/7} = 0.25286$
0.7	0.362	$(\Phi)^{-14.67/7} = 0.3648$
0.8	0.484	$(\Phi)^{-10.67/7} = 0.48033$
0.9	0.619	$(\Phi)^{-7/7} = 0.618033987$
1.2	1.098	$(\Phi)^{1.33/7} = 1.09599$
3.0	6.551	$(\Phi)^{27.33/7} = 6.547069$
3.5	8.937	$(\Phi)^{31.67/7} = 8.819$
4.0	11.749	$(\Phi)^{36/7} = 11.879$
4.5	15.023	$(\Phi)^{40/7} = 15.639$
6.0	27.728	$(\Phi)^{48/7} = 27.105$
7.0	38.919	$(\Phi)^{53/7} = 38.22389$

x	$\int_{0.3}^{x} \frac{\varphi'_0}{x^{3/2}\varphi_0^{1/2}}dx - \int_{0.3}^{x} \frac{\varphi_0^{1/2}}{x^{5/2}}dx$	$(\Phi)^{n/7}$
0.4	1.552	$(\Phi)^{6.33/7} = 1.5455$
0.5	2.377	$(\Phi)^{12.67/7} = 2.388$
0.6	2.863	$(\Phi)^{15.33/7} = 2.8693$
0.7	3.195	$(\Phi)^{17/7} = 3.2176$
0.8	3.412	$(\Phi)^{18/7} = 3.4466$
1.0	3.701	$(\Phi)^{19/7} = 3.6919$
1.2	3.872	$(\Phi)^{19.67/7} = 3.8650$
1.6	4.056	$(\Phi)^{20.33/7} = 4.0463$
2.0	4.152	$(\Phi)^{20.67/7} = 4.1401$
12-18	4.331	$(\Phi)^{21.33/7} = 4.3342$
20-34	4.332	“ “

x	$\int_{0.3}^{x} \left(\frac{\varphi_0}{x}\right)^{3/2} dx$	$(\Phi)^{n/7}$
0.4	0.282	$(\Phi)^{-18.33/7} = 0.28356$
0.5	0.452	$(\Phi)^{-11.67/7} = 0.4484$
0.6	0.562	$(\Phi)^{-8.33/7} = 0.56390$
0.8	0.693	$(\Phi)^{-5.33/7} = 0.69306$
1.2	0.811	$(\Phi)^{-3/7} = 0.8136$
1.8	0.872	$(\Phi)^{-2/7} = 0.87154$
3.5	0.911	$(\Phi)^{-1.33/7} = 0.9124$
4.0	0.913	“ “
4.5	0.915	“ “
5.0	0.916	“ “

In this chapter, we have utilized the following notations:

a = Bohr’s radius = 5.28×10^{-9} cm.
$\mu =$ measure unit of length (see eq.(2)).
$x =$ radius vector in unit μ.
$x_0 =$ radius of the atom.
$k =$ constant for the calculation of the potential (eqs.(7) and (8)).

The two relativistic eigenfunctions F and G (eqs.(10) and (11)), for $x < 0.3$, are given by:

$$G = x^{-\beta}\left[u_0 + u_1 x + u_2 x^2 + ...\right]; \qquad F = \frac{\beta}{\alpha Z} x^{-\beta}\left[v_0 + v_1 x + v_2 x^2 + ...\right].$$

For $x > 0.3$ we have that:

$$G = U / x; \qquad U = KR \sin\Theta .$$

For $x > x_0$ we have that:

$$U = B\sqrt[4]{x}\sin\left(C\sqrt{x} + D\right).$$

In the non-relativistic area G coincides with the Schroedinger's eigenfunction ψ.

3. On some equations concerning the p-Adic and Adelic Quantum Mechanics and the free relativistic particle. [2] [3]

According to the Weyl quantization, any function $f(k, x)$, of classical canonical variables k and x, which has the Fourier transform $\tilde{f}(\alpha, \beta)$ becomes a self-adjoint operator in $L_2(R^D)$ in the following way:

$$\tilde{f}(\hat{k}, \hat{x}) = \int \chi_\infty\left(-\alpha\hat{x} - \beta\hat{k}\right)\tilde{f}(\alpha, \beta) d^D\alpha d^D\beta . \quad (36)$$

Evolution of the elements $\Psi(x,t)$ of $L_2(R^D)$ is usually given by the Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = H\left(\hat{k}, x\right)\Psi(x,t), \quad (37)$$

where H is a Hamiltonian and $\hat{k}_j = -i\hbar\frac{\partial}{\partial x_j}$. Besides this differential equation there is also the following integral form:

$$\psi(x'', t'') = \int \mathcal{K}_\infty(x'', t''; x', t')\psi(x', t') d^D x', \quad (38)$$

where $\mathcal{K}_\infty(x'', t''; x', t')$ is the kernel of the unitary representation of the evolution operator $U_\infty(t'', t')$ and is postulated by Feynman to be a path integral

$$\mathcal{K}_\infty(x'', t''; x', t') = \int_{(x', t')}^{(x'', t'')} \chi_\infty(-S[q])\mathcal{D}q, \quad (39)$$

where functional $S[q] = \int_{t'}^{t''} L(\dot{q}, q, t)dt$ is the action for a path $q(t)$ in the classical Lagrangian $L(\dot{q}, q, t)$, and $x'' = q(t'')$, $x' = q(t')$ are the end points with the notation $x = (x_1, x_2, ..., x_D)$ and $q = (q_1, q_2, ..., q_D)$.

According to the Weyl approach to quantization, canonical non-commutativity in p-adic case should be introduced by operators $(h = 1)$

$$\hat{Q}_p(\alpha)\psi_p(x) = \chi_p(-\alpha x)\psi_p(x), \quad \hat{K}_p(\beta)\psi_p(x) = \psi_p(x + \beta) \quad (40)$$

which satisfy

$$\hat{Q}_p(\alpha)\hat{K}_p(\beta)=\chi_p(\alpha\beta)\hat{K}_p(\beta)\hat{Q}_p(\alpha), \quad (41)$$

where $\chi_p(u)=\exp(2\pi i\{u\}_p)$ is additive character on the field of p-adic numbers Q_p and $\{u\}_p$ is the fractional part of $u\in Q_p$. Let $\hat{x}$ and $\hat{k}$ be some operators of position x and momentum k, respectively. Let us define operators $\chi_v(\alpha\hat{x})$ and $\chi_v(\beta\hat{k})$ by formulas

$$\chi_v(\alpha\hat{x})\chi_v(ax)=\chi_v(\alpha x)\chi_v(ax), \quad \chi_v(\beta\hat{k})\chi_v(bk)=\chi_v(\beta k)\chi_v(bk), \quad (42)$$

where index v denotes real $(v=\infty)$ and any p-adic case, i.e. $v=\infty,2,...,p,...$ taking into account all non-trivial and inequivalent valuations on Q. It is evident that these operators also act on a function $\psi_v(x)$, which has the Fourier transform $\tilde{\psi}(k)$, in the following way:

$$\chi_v(-\alpha\hat{x})\psi_v(x)=\chi_v(-\alpha\hat{x})\int\chi_v(-kx)\tilde{\psi}(k)d^Dk=\chi_v(-\alpha x)\psi_v(x), \quad (43)$$
$$\chi_v(-\beta\hat{k})\psi_v(x)=\int\chi_v(-\beta k)\chi_v(-kx)\tilde{\psi}(k)d^Dk=\psi_v(x+\beta), \quad (44)$$

Comparing (40) with (43) and (44) we conclude that $\hat{Q}_p(\alpha)=\chi_p(-\alpha\hat{x})$, $\hat{K}_p(\beta)=\chi_p(-\beta\hat{k})$. Thus we have the following group relations concerning the p-adic cases:

$$\chi_v(-\alpha_i\hat{x}_i)\chi_v(-\beta_j\hat{k}_j)=\chi_v(\alpha_i\beta_j\delta_{ij})\chi_v(-\beta_j\hat{k}_j)\chi_v(-\alpha_i\hat{x}_i), \quad (45)$$
$$\chi_v(-\alpha_i\hat{x}_i)\chi_v(-\alpha_j\hat{x}_j)=\chi_v(-\alpha_j\hat{x}_j)\chi_v(-\alpha_i\hat{x}_i), \quad (46)$$
$$\chi_v(-\beta_i\hat{k}_i)\chi_v(-\beta_j\hat{k}_j)=\chi_v(-\beta_j\hat{k}_j)\chi_v(-\beta_i\hat{k}_i). \quad (47)$$

One can introduce the unitary operator

$$W_v(\alpha\hat{x},\beta\hat{k})=\chi_v\left(\frac{1}{2}\alpha\beta\right)\chi_v(-\beta\hat{k})\chi_v(-\alpha\hat{x}), \quad (48)$$

which satisfies the Weyl relation

$$W_v(\alpha\hat{x},\beta\hat{k})W_v(\alpha'\hat{x},\beta'\hat{k})=\chi_v\left(\frac{1}{2}(\alpha\beta'-\alpha'\beta)\right)W_v((\alpha+\alpha')\hat{x},(\beta+\beta')\hat{k}) \quad (49)$$

and is unitary representation of the Heisenberg-Weyl group. Recall that this group consists of the elements (z,η) with the group product

$$(z,\eta)\cdot(z',\eta')=\left(z+z',\eta+\eta'+\frac{1}{2}B(z,z')\right), \quad (50)$$

where $z=(\alpha,\beta)\in Q_v\times Q_v$ and $B(z,z')=\alpha\beta'-\beta\alpha'$ is a skew-symmetric bilinear form on the phase space. Using operator $W_v(\alpha\hat{x},\beta\hat{k})$ one can generalize Weyl formula for quantization (36) and it reads

$$\hat{f}_v(\hat{k},\hat{x})=\int W_v(\alpha\hat{x},\beta\hat{k})\tilde{f}_v(\alpha,\beta)d^D\alpha d^D\beta. \quad (51)$$

It is worth noting that equation (44) suggests to introduce

$$\{\beta\hat{k}\}_p^n \psi_p(x) = \int \{\beta k\}_p^n \chi_p(-kx) \tilde{\psi}_p(k) d^D k \quad (52)$$

which may be regarded as a new kind of the p-adic pseudodifferential operator. Also equation (45) suggests a p-adic analogue of the Heisenberg algebra in the form $(h=1)$

$$\{\alpha_i \hat{x}_i\}_p \{\beta_j \hat{k}_j\}_p - \{\beta_j \hat{k}_j\}_p \{\alpha_i \hat{x}_i\}_p = -\frac{i}{2\pi} \delta_{ij} \{\alpha\beta\}_p . \quad (53)$$

As a basic instrument to treat dynamics of a p-adic quantum model is natural to take the kernel $\mathcal{K}_p(x'',t'';x',t')$ of the evolution operator $U_p(t'',t')$. This kernel obtains by generalization of its real analogue starting from (38) and (39), i.e.

$$\psi_v(x'',t'') = \int \mathcal{K}_v(x'',t'';x',t') \psi_v(x',t') d^D x' , \quad (54)$$

and

$$\mathcal{K}_v(x'',t'';x',t') = \int_{(x',t')}^{(x'',t'')} \chi_v\left(-\int_{t'}^{t''} L(\dot{q},q,t) dt \right) \mathcal{D}q . \quad (55)$$

The p-adic quantum mechanics is given by a triple

$$\left(L_2(Q_p), W_p(z), U_p(t)\right), \quad (56)$$

where $W_p(z)$ corresponds to our $W_p(\alpha\hat{x},\beta\hat{k})$.
Starting from (54) one can easily derive the following three general properties:

$$\int \mathcal{K}_v(x'',t'';x,t) \mathcal{K}_v(x,t;x',t') dx = \mathcal{K}_v(x'',t'';x',t'), \quad (57)$$

$$\int \overline{\mathcal{K}_v}(x'',t'';x',t') \mathcal{K}_v(y,t'';x',t') dx' = \delta_v(x''-y), \quad (58)$$

$$\mathcal{K}_v(x'',t'';x',t'') = \lim_{t'\to t''} \mathcal{K}_v(x'',t'';x',t') = \delta_v(x''-x') . \quad (59)$$

Quantum fluctuations lead to deformations of classical trajectory and any quantum history may be presented as $q(t) = x(t) + y(t)$, where $y' = y(t') = 0$ and $y'' = y(t'') = 0$. For Lagrangians $L(\dot{q},q,t)$ which are quadratic polynomials in $\dot{q}$ and q, the corresponding Taylor expansion of $S[q]$ around classical path $x(t)$ is

$$S[q] = S[x] + \frac{1}{2!}\delta^2 S[x] = S[x] + \frac{1}{2}\int_{t'}^{t''} \left(\dot{y}\frac{\partial}{\partial \dot{q}} + y\frac{\partial}{\partial q} \right)^2 L(\dot{q},q,t) dt , \quad (60)$$

where we have used $\delta S[x] = 0$. Hence we get

$$\mathcal{K}_v(x'',t'';x',t')=\chi_v\left(-\frac{1}{h}\bar{S}(x'',t'';x',t')\right)\times\int_{(y'\to 0,t')}^{(y''\to 0,t'')}\chi_v\left[-\frac{1}{2h}\int_{t'}^{t''}\left(\dot{y}\frac{\partial}{\partial\dot{q}}+y\frac{\partial}{\partial q}\right)^2 L(\dot{q},q,t)dt\right]\mathcal{D}y\,, \quad (61)$$

where $\bar{S}(x'',t'';x',t')=S[x]$. From (61) follows that $\mathcal{K}_v(x'',t'';x',t')$ has the form

$$\mathcal{K}_v(x'',t'';x',t')=N_v(t'',t')\chi_v\left(-\frac{1}{h}\bar{S}(x'',t'';x',t')\right), \quad (62)$$

where $N_v(t'',t')$ does not depend on end points x'' and x'. To calculate $N_v(t'',t')$ one can use (57) and (58). Then one obtains that v-adic kernel $\mathcal{K}_v(x'',t'';x',t')$ of the unitary evolution operator for one-dimensional systems with quadratic Lagrangians has the form

$$\mathcal{K}_v(x'',t'';x',t')=\lambda_v\left(-\frac{1}{2h}\frac{\partial^2}{\partial x''\partial x'}\bar{S}(x'',t'';x',t')\right)\times\left|\frac{1}{h}\frac{\partial^2}{\partial x''\partial x'}\bar{S}(x'',t'';x',t')\right|_v^{1/2}\chi_v\left(-\frac{1}{h}\bar{S}(x'',t'';x',t')\right), \quad (63)$$

For practical considerations, we define adelic path integral in the form

$$\mathcal{K}_{A}(x'',t'';x',t')=\prod_v\int_{(x_v',t_v')}^{(x_v'',t_v'')}\chi_v\left(-\frac{1}{h}\int_{t_v'}^{t_v''}L(\dot{q}_v,q_v,t_v)dt_v\right)\mathcal{D}q_v\,. \quad (64)$$

As an adelic Lagrangian one understands an infinite sequence

$$L_A(\dot{q},q,t)=\left(L(\dot{q}_\infty,q_\infty,t_\infty),L(\dot{q}_2,q_2,t_2),L(\dot{q}_3,q_3,t_3),\ldots,L(\dot{q}_p,q_p,t_p),\ldots\right), \quad (65)$$

where $\left|L(\dot{q}_p,q_p,t_p)\right|_p\le 1$ for all primes p but a finite set **S** of them.

As an illustration of p-adic quantum-mechanical models, now we describe the following one-dimensional system where the quadratic Lagrangians is considered: a free relativistic particle $L(\dot{q},q)=-mc^2\sqrt{\dot{q}_\mu\dot{q}^\mu}$.

3.1 Free relativistic electron in p-adic quantum mechanics. [3]

In the Vladimirov-Volovich formulation one-dimensional p-adic quantum mechanics is a triple

$$\left(L_2(Q_p),W_p(z),U_p(t)\right), \quad (66)$$

where $L_2(Q_p)$ is the Hilbert space of complex-valued functions of p-adic variables, $W_p(z)$ is a unitary representation of the Heisenberg-Weyl group on $L_2(Q_p)$, and $U_p(t)$ is an evolution operator on $L_2(Q_p)$.

$U_p(t)$ is an integral operator

$$U_p(t)\psi_p(x)=\int_{Q_p}K_p(x,t;y,0)\psi_p(y)dy \quad (67)$$

whose kernel is given by the Feynman path integral

$$K_p(x,t;y,0)=\int \chi_p\left(-\frac{1}{h}S[q]\right)\mathcal{D}q=\int \chi_p\left(-\frac{1}{h}\int_0^t L(q,\dot{q})dt\right)\prod_t dq(t), \quad (68)$$

where h is the Planck constant. For quadratic classical actions $\bar{S}(x,t;y;0)$ the solution (68) becomes

$$K_p(x,t;y,0)=\lambda_p\left(-\frac{1}{2h}\frac{\partial^2\bar{S}}{\partial x\partial y}\right)\left|\frac{1}{h}\frac{\partial^2\bar{S}}{\partial x\partial y}\right|_p^{1/2}\chi_p\left(-\frac{1}{h}\bar{S}(x,t;y,0)\right). \quad (69)$$

Expression (69) has the same form as that one in ordinary quantum mechanics.
Thence, we can rewrite the eq. (68) also:

$$\lambda_p\left(-\frac{1}{2h}\frac{\partial^2\bar{S}}{\partial x\partial y}\right)\left|\frac{1}{h}\frac{\partial^2\bar{S}}{\partial x\partial y}\right|_p^{1/2}\chi_p\left(-\frac{1}{h}\bar{S}(x,t;y,0)\right)=\int \chi_p\left(-\frac{1}{h}S[q]\right)\mathcal{D}q=\int \chi_p\left(-\frac{1}{h}\int_0^t L(q,\dot{q})dt\right)\prod_t dq(t).$$

(69b)

For a particular physical system, p-adic eigenfunctions are subject of the spectral problem

$$U_p(t)\psi_p^{(\alpha)}(x)=\chi_p(\alpha t)\psi_p^{(\alpha)}(x). \quad (70)$$

The usual action for a free relativistic electron

$$S=-mc^2\int_{\tau_1}^{\tau_2}d\tau\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \quad (71)$$

is nonlinear and so unsuitable for quantum-mechanical investigations. However, a free relativistic electron can be treated as a system with the constraint $\eta_{\mu\nu}k^\mu k^\nu+m^2c^2=k^2+m^2c^2=0$, which leads to the canonical Hamiltonian (with the Lagrange multiplier N)

$$H_c=N\left(k^2+m^2c^2\right), \quad (72)$$

and to the Lagrangian

$$L=\dot{x}_\mu k^\mu-H_c=\frac{\dot{x}^2}{4N}-m^2c^2N, \quad (73)$$

where $\dot{x}_\mu=\partial H_c/\partial k^\mu=2k_\mu$ and $\dot{x}^2=\dot{x}^\mu\dot{x}_\mu$. Instead of (71), the corresponding action for quantum treatment of a free relativistic electron is

$$S=\int_{\tau_1}^{\tau_2}d\tau\left(\frac{\dot{x}^2}{4N}-m^2c^2N\right). \quad (74)$$

From (74) it follows the classical trajectory

$$\bar{x}^\mu=\frac{x_2^\mu-x_1^\mu}{\tau_2-\tau_1}\tau+\frac{x_1\tau_2-x_2\tau_1}{\tau_2-\tau_1} \quad (75)$$

and the classical action

$$\bar{S}(x_2,T;x_1,0)=\frac{(x_2-x_1)^2}{4T}-m^2c^2T\,, \quad (76)$$

where $T=N(\tau_2-\tau_1)$.

All the above expressions from (72) to (76) are valid in the real case and according to p-adic analysis they have place in the p-adic one. Note that the classical action (76) can be presented in the form

$$\bar{S}=\left[-\frac{(x_2^0-x_1^0)^2}{4T}-\frac{m^2c^2T}{4}\right]+\left[\frac{(x_2^1-x_1^1)^2}{4T}-\frac{m^2c^2T}{4}\right]+\left[\frac{(x_2^2-x_1^2)^2}{4T}-\frac{m^2c^2T}{4}\right]+$$

$$+\left[\frac{(x_2^3-x_1^3)^2}{4T}-\frac{m^2c^2T}{4}\right]=\bar{S}^0+\bar{S}^1+\bar{S}^2+\bar{S}^3 \quad (77)$$

which is quadratic in x_2^μ and x_1^μ $(\mu=0,1,2,3)$. Due to (69) and (77), the corresponding quantum-mechanical propagator may be written as product

$$K_p(x_2,T;x_1,0)=\prod_{\mu=0}^{3}K_p^{(\mu)}(x_2^\mu,T;x_1^\mu,0), \quad (78)$$

$$K_p^{(\mu)}(x_2^\mu,T;x_1^\mu,0)=\lambda_p\left((-1)^{\delta_0^\mu}4hT\right)|2hT|_p^{-1/2}\times\chi_p\left[-\frac{1}{h}(-1)^{\delta_0^\mu}\frac{(x_2^\mu-x_1^\mu)^2}{4T}+\frac{1}{h}\frac{m^2c^2T}{4}\right], \quad (79)$$

where $\delta_0^\mu=1$ if $\mu=0$ and 0 otherwise. We note that the eq. (78) can be rewritten also:

$$K_p(x_2,T;x_1,0)=\prod_{\mu=0}^{3}\lambda_p\left((-1)^{\delta_0^\mu}4hT\right)|2hT|_p^{-1/2}\times\chi_p\left[-\frac{1}{h}(-1)^{\delta_0^\mu}\frac{(x_2^\mu-x_1^\mu)^2}{4T}+\frac{1}{h}\frac{m^2c^2T}{4}\right]. \quad (80)$$

Among all possible eigenstates which satisfy eq. (70), function $\Omega(|x|_p)$ defined by the following expression

$$\Omega(|u|_p)=1,\ |u|_p\le 1; \qquad \Omega(|u|_p)=0,\ |u|_p>1, \quad (81)$$

plays a central role in p-adic and adelic quantum mechanics. Therefore, let us first show existence of Ω-eigenfunction for the above relativistic electron. In fact, we have now 1+3 dimensional problem and the corresponding integral equation is

$$\int_{Q_p^4}K_p(x,T;y,0)\Omega(|y|_p)d^4y=\Omega(|x|_p), \quad (\alpha=0), \quad (82)$$

where $|u|_p=\max_{0\le\mu\le3}\{|u^\mu|_p\}$ is p-adic norm of $u\in Q_p^4$, and

$$K_p(x,T;y,0)=\frac{\lambda_p^2(4hT)}{|2hT|_p^2}\chi_p\left(-\frac{(x-y)^2}{4hT}+\frac{m^2c^2T}{h}\right). \quad (83)$$

Thence, we have that

$$\int_{Q_p^4} \frac{\lambda_p^2(4hT)}{|2hT|_p^2} \chi_p\left(-\frac{(x-y)^2}{4hT}+\frac{m^2c^2T}{h}\right)\Omega\left(|y|_p\right)d^4y=\Omega\left(|x|_p\right). \quad (83b)$$

Eq. (82), rewritten in a more explicit form, reads

$$\frac{\lambda_p^2(4hT)}{|2hT|_p^2} \chi_p\left(\frac{m^2c^2T}{h}-\frac{x^2}{4hT}\right)\int_{Z_p} \chi_p\left(\frac{(y^0)^2}{4hT}-\frac{x^0y^0}{2hT}\right)dy^0 \times \prod_{i=1}^{3}\int_{Z_p} \chi_p\left(-\frac{(y^i)^2}{4hT}+\frac{x^iy^i}{2hT}\right)dy^i=\Omega\left(|x|_p\right). \quad (84)$$

Using lower part of the Gauss integral

$$\int_{|x|_p \le p^\nu} \chi_p\left(\alpha x^2+\beta x\right)dx = p^\nu \Omega\left(p^\nu |\beta|_p\right), \quad |\alpha|_p \le p^{-2\nu};$$

$$\int_{|x|_p \le p^\nu} \chi_p\left(\alpha x^2+\beta x\right)dx = \lambda_p(\alpha)|2\alpha|_p^{-1/2} \chi_p\left(-\frac{\beta^2}{4\alpha}\right)\Omega\left(p^{-\nu}\left|\frac{\beta}{2\alpha}\right|_p\right), \quad |4\alpha|_p > p^{-2\nu}, \quad (84b)$$

to calculate integrals in (84) for each coordinate y^μ $(\mu = 0,\ldots,3)$, we obtain

$$\chi_p\left(\frac{m^2c^2T}{h}\right)\prod_{\mu=0}^{3}\Omega\left(|x^\mu|_p\right)=\Omega\left(|x|_p\right), \quad |hT|_p < 1. \quad (85)$$

Since $\Pi_{\mu=0}^3 \Omega\left(|x^\mu|_p\right)=\Omega\left(|x|_p\right)$ is an identity, an equivalent assertion to (85) is

$$\left|\frac{m^2c^2T}{h}\right|_p \le 1, \quad |hT|_p < 1. \quad (86)$$

Applying also the upper part of (84b) to (84), we have

$$\frac{\lambda_p^2(4hT)}{|2hT|_p^2} \chi_p\left(\frac{m^2c^2T}{h}-\frac{x^2}{4hT}\right)\prod_{\mu=0}^{3}\Omega\left(\left|\frac{x^\mu}{2hT}\right|_p\right)=\Omega\left(|x|_p\right), \quad |4hT|_p \ge 1, \quad (87)$$

what is satisfied only for $p \ne 2$. Namely, (87) becomes an equality if conditions

$$\left|\frac{m^2c^2T}{h}\right|_p \le 1, \quad |hT|_p = 1, \quad p \ne 2, \quad (88)$$

take place. Thus, we obtained eigenstates

$$\psi_p(x,T)=\Omega\left(|x|_p\right), \quad \left|\frac{m^2c^2T}{h}\right|_p \le 1, \quad |hT|_p \le 1, \quad p \ne 2, \quad (89)$$

$$\psi_p(x,T)=\Omega(|x|_2),\ \left|\frac{m^2c^2T}{h}\right|_2 \le 1,\quad |hT|_2<1, \qquad (90)$$

which are invariant under $U_p(t)$ transformation.
We have also Ω-function in eigenstates

$$\psi_p(x,T)=\chi_p\left(\frac{m^2c^2}{h}T\right)\Omega\left(p^\nu|x|_p\right),\quad \nu\in Z,\quad |hT|_p<p^{-2\nu}. \quad (91)$$

This can be shown in the way similar to the previous case with $\psi_p(x,T)=\Omega\left(|x|_p\right)$. The eigenstates without Ω-functions are as follows:

$$\psi_p(x,T)=\chi_p\left(\frac{m^2c^2+k^2}{h}T\right)\chi_p\left(-\frac{kx}{h}\right), \quad (92)$$

where $k^2=-k^0k^0+k^ik^i$, and $kx=-k^0x^0+k^ix^i$. Note that $\left(m^2c^2+k^2\right)T=H_c\tau$.

3.2 p-Adic and Adelic strings. [4]

Recall that quantum amplitudes defined by means of path integral may be symbolically presented as

$$A(K)=\int A(X)\chi\left(-\frac{1}{h}S[X]\right)\mathcal{D}X, \quad (93)$$

where K and X denote classical momenta and configuration space, respectively. Here, $\chi(a)$ is an additive character, $S[X]$ is a classical action and h is the Planck constant.
Now we consider simple *p-adic and adelic bosonic string amplitudes* based on the functional integral (93). The scattering of two real bosonic strings in 26-dimensional space-time at the tree level can be described in terms of the path integral in 2-dimensional quantum field theory formalism as follows:

$$A_\infty(k_1,\dots,k_4)=g_\infty^2\int\mathcal{D}X\exp\left(\frac{2\pi i}{h}S_0[X]\right)\times\prod_{j=1}^{4}\int d^2\sigma_j\exp\left(\frac{2\pi i}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right), \quad (94)$$

where $\mathcal{D}X=\mathcal{D}X^0(\sigma,\tau)\mathcal{D}X^1(\sigma,\tau)\dots\mathcal{D}X^{25}(\sigma,\tau)$, $d^2\sigma_j=d\sigma_jd\tau_j$ and

$$S_0[X]=-\frac{T}{2}\int d^2\sigma\partial_\alpha X^\mu\partial^\alpha X_\mu \quad (95)$$

with $\alpha=0,1$ and $\mu=0,1,\dots,25$. Using the usual procedure one can obtain the crossing symmetric Veneziano amplitude

$$A_\infty(k_1,\dots,k_4)=g_\infty^2\int_R|x|_\infty^{k_1k_2}|1-x|_\infty^{k_2k_3}dx \quad (96)$$

and similarly the Virasoro-Shapiro one for closed bosonic strings. As p-adic Veneziano amplitude, it was postulated p-adic analogue of (96), i.e.

$$A_p(k_1,...,k_4) = g_p^2 \int_{Q_p} |x|_p^{k_1 k_2} |1-x|_p^{k_2 k_3} dx, \quad (97)$$

where only the string world sheet (parameterized by x) is p-adic. Expressions (96) and (97) are Gel'fand-Graev beta functions on R and Q_p, respectively. We take p-adic analogue of (94), i.e.

$$A_p(k_1,...,k_4) = g_p^2 \int \mathcal{D}X \chi_p\left(-\frac{1}{h} S_0[X]\right) \times \prod_{j=1}^{4} \int d^2\sigma_j \chi_p\left(-\frac{1}{h} k_\mu^{(j)} X^\mu(\sigma_j, \tau_j)\right), \quad (98)$$

to be p-adic string amplitude, where $\chi_p(u) = \exp(2\pi i \{u\}_p)$ is p-adic additive character and $\{u\}_p$ is the fractional part of $u \in Q_p$. In (98), all space-time coordinates X_μ, momenta k_i and world sheet (σ, τ) are p-adic.
Adelic string amplitude is product of real and all p-adic amplitudes, i.e.

$$A_A(k_1,...,k_4) = A_\infty(k_1,...,k_4) \prod_p A_p(k_1,...,k_4). \quad (99)$$

In the case of the Veneziano amplitude and $(\sigma_i, \tau_j) \in \mathrm{A}(S) \times \mathrm{A}(S)$, where $\mathrm{A}(S)$ is defined in the following equation (i.e. the set of all adeles A, where A has the structure of a topological ring)

$$A = \bigcup_S \mathrm{A}(S), \quad \mathrm{A}(S) = R \times \prod_{p \in S} Q_p \times \prod_{p \notin S} Z_p, \quad (100)$$

we have

$$A_A(k_1,...,k_4) = g_\infty^2 \int_R |x|_\infty^{k_1 k_2} |1-x|_\infty^{k_2 k_3} dx \times \prod_{p \in S} g_p^2 \prod_{j=1}^{4} \int d^2\sigma_j \times \prod_{p \notin S} g_p^2. \quad (101)$$

4. Mathematical connections.

Now we show some interesting mathematical connections that we have obtained between various equations described above.
We note that the eqs. (31), (34) and (34b) of **Section 2**, and the eqs. (61), (64), (69b), (84), (93) and (98) of **Section 3** can be related. Thence, we have the possible mathematical connection between the relativistic equations of quantum mechanics, thence some equations concerning the free relativistic electron, the free relativistic electron in p-adic quantum mechanics and p-adic bosonic string amplitudes. Indeed, we have the following principal connections between the eqs. (31), (69b), (84), (98) and (34b); and between eqs. (34), (69b), (84), (98) and (34b):

$$\Theta(0,3) + \left[\int \sqrt{g}\,dx - \frac{1}{32}\int \frac{g'^2}{g^{5/2}} dx - \frac{1}{8}\frac{g'}{g^{3/2}} + \frac{\gamma^2}{2}\left(\int \frac{g' dx}{x\sqrt{g}} + \int g^{3/2} dx\right)\right]_{0,3}^{x} \Rightarrow$$

$$\Rightarrow \lambda_p\left(-\frac{1}{2h}\frac{\partial^2 \bar{S}}{\partial x \partial y}\right)\left|\frac{1}{h}\frac{\partial^2 \bar{S}}{\partial x \partial y}\right|_p^{1/2} \chi_p\left(-\frac{1}{h}\bar{S}(x,t;y,0)\right) =$$

$$= \int \chi_p \left(-\frac{1}{h} S[q] \right) \mathcal{D}q = \int \chi_p \left(-\frac{1}{h} \int_0^t L(q,\dot{q}) dt \right) \prod_t dq(t) \Rightarrow$$

$$\Rightarrow \frac{\lambda_p^2 (4hT)}{|2hT|_p^2} \chi_p \left(\frac{m^2 c^2 T}{h} - \frac{x^2}{4hT} \right) \int_{Z_p} \chi_p \left(\frac{(y^0)^2}{4hT} - \frac{x^0 y^0}{2hT} \right) dy^0 \times \prod_{i=1}^{3} \int_{Z_p} \chi_p \left(-\frac{(y^i)^2}{4hT} + \frac{x^i y^i}{2hT} \right) dy^i = \Omega(|x|_p) \Rightarrow$$

$$\Rightarrow A_p(k_1,...,k_4) = g_p^2 \int \mathcal{D}X \chi_p \left(-\frac{1}{h} S_0[X] \right) \times \prod_{j=1}^{4} \int d^2\sigma_j \chi_p \left(-\frac{1}{h} k_\mu^{(j)} X^\mu (\sigma_j, \tau_j) \right) \Rightarrow$$

$$\Rightarrow -\int d^{26}x \sqrt{g} \left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu \left(|F_2|^2 \right) \right]. \quad (102)$$

$$\Theta(0,3) + A^{1/2} \left\{ \int_{0,3}^{x} \left(\frac{\varphi_0}{x} \right)^{1/2} dx - \frac{k}{2} \int_{0,3}^{x} \frac{\eta}{(x\varphi_0)^{1/2}} dx + \frac{1}{2Zx_0} \int_{0,3}^{x} \left(\frac{x}{\varphi_0} \right)^{1/2} dx \right\} +$$

$$- A^{-1/2} \left\{ \frac{1}{32} \left[\int_{0,3}^{x} \frac{x^{1/2} \varphi'^2_0}{\varphi_0^{5/2}} dx + \int_{0,3}^{x} \frac{dx}{\varphi_0^{1/2} x^{3/2}} - 2 \int_{0,3}^{x} \frac{\varphi'_0}{x^{1/2} \varphi_0^{3/2}} dx \right] + \frac{1}{8} \left(\frac{x^{1/2} \varphi'_0}{\varphi_0^{3/2}} - \frac{1}{(x\varphi_0)^{1/2}} \right)_{0,3}^{x} \right\} +$$

$$+ \frac{\gamma^2}{2} \left\{ A^{1/2} \left[\int_{0,3}^{x} \frac{\varphi'_0}{x^{3/2} \varphi_0^{1/2}} dx - \int_{0,3}^{x} \frac{\varphi_0^{1/2}}{x^{5/2}} dx \right] + A^{3/2} \int_{0,3}^{x} \left(\frac{\varphi_0}{x} \right)^{3/2} dx \right\} \Rightarrow$$

$$\Rightarrow \lambda_p \left(-\frac{1}{2h} \frac{\partial^2 \bar{S}}{\partial x \partial y} \right) \left| \frac{1}{h} \frac{\partial^2 \bar{S}}{\partial x \partial y} \right|_p^{1/2} \chi_p \left(-\frac{1}{h} \bar{S}(x,t;y,0) \right) =$$

$$= \int \chi_p \left(-\frac{1}{h} S[q] \right) \mathcal{D}q = \int \chi_p \left(-\frac{1}{h} \int_0^t L(q,\dot{q}) dt \right) \prod_t dq(t) \Rightarrow$$

$$\Rightarrow \frac{\lambda_p^2 (4hT)}{|2hT|_p^2} \chi_p \left(\frac{m^2 c^2 T}{h} - \frac{x^2}{4hT} \right) \int_{Z_p} \chi_p \left(\frac{(y^0)^2}{4hT} - \frac{x^0 y^0}{2hT} \right) dy^0 \times \prod_{i=1}^{3} \int_{Z_p} \chi_p \left(-\frac{(y^i)^2}{4hT} + \frac{x^i y^i}{2hT} \right) dy^i = \Omega(|x|_p) \Rightarrow$$

$$\Rightarrow A_p(k_1,...,k_4) = g_p^2 \int \mathcal{D}X \chi_p \left(-\frac{1}{h} S_0[X] \right) \times \prod_{j=1}^{4} \int d^2\sigma_j \chi_p \left(-\frac{1}{h} k_\mu^{(j)} X^\mu (\sigma_j, \tau_j) \right) \Rightarrow$$

$$\Rightarrow -\int d^{26}x \sqrt{g} \left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x (-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu \left(|F_2|^2 \right) \right]. \quad (103)$$

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